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Fuzzy roughness via ideals

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Abstract. In this paper, we join the notion of fuzzy ideal to the notion of fuzzy approximation space to define the notion of fuzzy ideal approximation spaces. We introduce the fuzzy ideal approximation interior operator $\mathrm{int}_\Phi^\lambda$ and the fuzzy ideal approximation closure operator cl_Φ^λ, and moreover, we define the fuzzy ideal approximation preinterior operator $\mathrm{pint}_\Phi^\lambda$ and the fuzzy ideal approximation preclosure operator $\mathrm{pcl}_\Phi^\lambda$ with respect to that fuzzy ideal defined on the fuzzy approximation space (X, R) associated with some fuzzy set $\lambda \in I^X$. Also, we define fuzzy separation axioms, fuzzy connectedness and fuzzy compactness in fuzzy approximation spaces and in fuzzy ideal approximation spaces as well, and prove the implications in between.

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1. Introduction

Pawlak ([21]) has defined the notion of rough sets referring to uncertainty of intelligent systems. In [21], there are more theoretical aspects of rough sets and its applications. An extended study of fuzzy Lie Algebras is found in [1]. An approximation space (X, R) is constructed from a universe set of objects and an equivalence relation on these objects. The boundary region between the lower approximation set A_R and the upper approximation set A^R of a set A in (X, R) described these rough sets. If the lower and the upper approximation sets are equal, then A is then an exact subset of X and there is no roughness. Many researchers studied the relationship between rough sets and topological spaces in [7, 18, 24] The notion of ideal in topological spaces was defined and studied in [13] and the notion of a fuzzy ideal was given in [23]. The local function of some subset in a topological space was defined and studied in [25]. Many studies have been published based on joining an ideal to a topological space as in [8, 9, 14–16]. Separation axioms with respect to an ideal were given in [2], and the notion of continuity via ideals was given in [3] while the notion of grills on a topological space was introduced by Choquet [6] and fuzzy grills on X was given in [4]. The concepts of ideals and grills have proved to be a powerful supporting as known with filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces, connectedness and compactness ([10, 14, 15, 22]). In [22], the authors defined and studied a typical topology associated naturally to the existing topology and a grill on a given topological space. Hatir and Jafari [10] defined new classes of sets and gave a new decomposition of continuity in terms of grills. In [20], the authors studied fuzzy soft separation axioms and fuzzy soft connectedness in fuzzy topological spaces in sense of Chang ([5]). In [17], the authors introduced some concepts in fuzzy ideal topological spaces. Graded fuzzy separation axioms were defined in [11], and by the way fuzzy approximation

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and fuzzy ideal approximation separation axioms will be defined in Section 4. Fuzzy approximation compactness and fuzzy ideal approximation compactness will be defined in Section 5. Fuzzy lower and fuzzy upper sets of a rough set were studied in [19].

In this paper, we joined the notion of fuzzy ideal ℓ with the fuzzy approximation space (X, R) associated with a fuzzy set λ, and defined fuzzy interior and fuzzy closure operators with respect to that fuzzy ideal. The local function $\Phi_\lambda(\mu)$ of some $\mu \in I^X$ with respect to that fuzzy ideal was a base in defining the related interior and closure operators. Separation axioms in fuzzy approximation spaces and in fuzzy ideal approximation spaces were defined and compared with examples to confirm the implications in between. Connectedness in fuzzy approximation spaces and in fuzzy ideal approximation spaces were defined and compared with examples to show the implications in between. Compactness in fuzzy approximation spaces and in fuzzy ideal approximation spaces were defined as well. All results studied in fuzzy ideal approximation spaces are directly proved if we changed to the fuzzy grill approximation spaces. The correspondence between fuzzy ideal and fuzzy grill was insured in [16]. Fuzzy approximation continuity and fuzzy ideal approximation continuity were introduced as well.

The motivation of Section 1 is to define the fuzzy approximation lower and upper sets, and then to define the fuzzy approximation interior and closure operators on a fuzzy approximation space. Through these fuzzy operators we defined fuzzy approximation separation axioms, fuzzy approximation connectedness and fuzzy approximation compactness. A generalization of these definitions is defined using a fuzzy ideal constructed on the fuzzy approximation space.

Through the paper, let X be a set of objects, I the closed unit interval $[0, 1]$ and $I_0 = (0, 1]$. I^X denotes all the fuzzy subsets of X, and $\lambda^c(x) = 1 - \lambda(x)\ \forall x \in X, \forall \lambda \in I^X$. A constant fuzzy set $\bar{t}$ for all $t \in I$ is defined by $\bar{t}(x) = t\ \forall x \in X$. Infimum and supremum of a fuzzy set $\lambda \in I^X$ are given as: $\inf \lambda = \bigwedge_{x\in X} \lambda(x)$ and $\sup \lambda = \bigvee_{x\in X} \lambda(x)$. If $f : X \to Y$ is a mapping, $\mu \in I^X$, $\nu \in I^Y$, then

$$(f(\mu))(y) = \bigvee_{x\in f^{-1}(y)} \mu(x)\ \forall y \in Y \text{ and } f^{-1}(\nu) = (\nu \circ f).$$

Assume a fuzzy relation $R : X \times X \to I$ is defined so that $R(x, x) = 1\ \ \forall x \in X$, $R(x, y) = R(y, x)\ \ \forall x, y \in X$ and $R(x, y) \geq (R(x, z) \wedge R(z, y))\ \forall x, y, z \in X$. That is, R is a fuzzy equivalence relation on X. (X, R) is called a fuzzy approximation space based on the fuzzy equivalence relation R on X.

Definition 1.1. For each $x \in X$, define a fuzzy coset $[x] : X \to I$ by:

$$[x](y) = R(x, y)\ \forall y \in X \tag{1}$$

All elements $y \in X$ with fuzzy relation value $R(x, y) > 0$ are elements having a membership value in the fuzzy coset $[x]$, and any element $y \in X$ with $R(x, y) = 0$ is not included in the fuzzy coset $[x]$. Any fuzzy coset $[x]$ surely include the element $x \in X$, and consequently $\bigvee_{z\in X} [x](z) = 1$ for all $x \in X$. Also, $\bigvee_{z\in X} [z](y) = 1\ \forall y \in X$ (i.e. $\bigvee_{z\in X} [z] = \bar{1}$). Clearly, if $R(x, y) > 0$, then the fuzzy cosets $[x], [y]$ (as fuzzy sets) are containing the same elements of X with some non zero membership values, and moreover if $[y](z) = 0$, then it must be that $[x](z) = 0$ whenever $R(x, y) > 0$. That is, any two fuzzy cosets are either two fuzzy sets containing the same elements of X with some non zero membership values or containing completely different elements of X with some non zero membership values. Strictly, in case of $I = \{0, 1\}$ it is a partitioning of X as usually known in the general case.

Note that: $[x] \neq \bar{0}\ \forall x \in X$ since there is at least $x \in X$ itself such that $[x](x) = 1$, while may be all elements $z \in X$ are given such that $[x](z) > 0\ \forall z \in X$. The fuzzy cosets could be such that $[x](x) = 1$ and $[x](z) = 0\ \forall z \neq x$, which means (X, R) is fuzzy partitioned into completely disjoint fuzzy cosets. Putting $I = \{0, 1\}$ as a crisp case, we get exactly the usual meaning of partitioning of a set X based on an ordinary equivalence relation R on X.

Recall that the fuzzy difference between two fuzzy sets was defined ([12]) as:

$$(\lambda \mathbin{\bar{\wedge}} \mu) = \begin{cases} \bar{0} & \text{if } \lambda \leq \mu, \\ \lambda \wedge \mu^c & \text{otherwise.} \end{cases} \tag{2}$$

Definition 1.2. Let $\lambda \in I^X$ and R a fuzzy equivalence relation on X and the fuzzy cosets are defined as in (1). Then, the fuzzy lower set λ_R, the fuzzy upper set λ^R and the fuzzy boundary region set λ^B are defined as follows:

$$\lambda_R(x) = \lambda(x) \wedge \Big(\bigvee_{\lambda^c(z)>0,\ z \neq x} [x](z) \Big)^c \quad \forall x \in X, \tag{3}$$

$$\lambda^R(x) = \lambda(x) \vee \bigvee_{\lambda(z)>0,\ z \neq x} [x](z) \quad \forall x \in X, \tag{4}$$

$$\lambda^B = \lambda^R \,\bar{\wedge}\, \lambda_R = \begin{cases} \overline{0} & \text{if } \lambda^R \leq \lambda_R \\ \lambda^R \wedge (\lambda_R)^c & \text{otherwise.} \end{cases} \tag{5}$$

λ_R, λ^R and λ^B are then called fuzzy lower, fuzzy upper and fuzzy boundary region sets associated with the fuzzy set λ in I^X and based on the fuzzy equivalence relation R in the fuzzy approximation space (X, R).

From (3) and (4), we get that $\lambda_R \leq \lambda \leq \lambda^R \ \forall \lambda \in I^X$. Whenever λ^R be so that $\lambda^R \leq \lambda_R$, we get that $\lambda = \lambda_R = \lambda^R$ and then from (5), we have $\lambda^B = \overline{0}$. Otherwise, $\lambda^B = \lambda^R \wedge (\lambda_R)^c$. The fuzzy accuracy $\alpha_R(\lambda)$ of approximation of the fuzzy set λ could be characterized numerically by $\alpha_R(\lambda) = \frac{\inf \lambda_R}{\sup \lambda^R}$, where $0 \leq \alpha_R(\lambda) \leq 1$. If $\alpha_R(\lambda) = 1$, then λ is crisp with respect to R ($\lambda_R = \lambda^R$ and λ is precise with respect to R), and otherwise, if $\alpha_R(\lambda) < 1$, λ is rough with respect to R (λ is vague with respect to R).

Lemma 1.1. *For any fuzzy set $\lambda \in I^X$ we get easily that:*

(1) $\overline{0}_R = \overline{0}^R = \overline{0}$ and $\overline{1}_R = \overline{1}^R = \overline{1}$,
(2) $(\lambda \vee \mu)_R \geq \lambda_R \vee \mu_R$, $(\lambda \wedge \mu)^R \leq \lambda^R \wedge \mu^R$,
(3) $\lambda \leq \mu$ implies that $\lambda_R \leq \mu_R$ and $\lambda^R \leq \mu^R$,
(4) $(\lambda \vee \mu)^R = \lambda^R \vee \mu^R$, $(\lambda \wedge \mu)_R = \lambda_R \wedge \mu_R$,
(5) $(\lambda^R)^c = (\lambda^c)_R$ and $(\lambda_R)^c = (\lambda^c)^R$
(6) $(\lambda_R)^R \geq (\lambda_R)_R = \lambda_R$, $(\lambda^R)_R \leq (\lambda^R)^R = \lambda^R$.

Associated with a fuzzy set λ in a fuzzy approximation space (X, R), it was defined a fuzzy interior operator $\mathrm{int}^\lambda_R : I^X \to I^X$ as follows:

$$\mathrm{int}^\lambda_R(\nu) = \lambda_R \wedge \nu_R \quad \forall \nu \neq \overline{1} \quad \text{and } \mathrm{int}^\lambda_R(\overline{1}) = \overline{1}. \tag{6}$$

Also, it was defined a fuzzy closure operator $\mathrm{cl}^\lambda_R : I^X \to I^X$ as follows:

$$\mathrm{cl}^\lambda_R(\nu) = (\lambda_R)^c \vee \nu^R \quad \forall \nu \neq \overline{0} \quad \text{and } \mathrm{cl}^\lambda_R(\overline{0}) = \overline{0}. \tag{7}$$

Recall that:

$$\mathrm{cl}^\lambda_R(\nu^R) = \mathrm{cl}^\lambda_R(\nu)\ \forall \nu \in I^X, \quad \mathrm{int}^\lambda_R(\nu_R) = \mathrm{int}^\lambda_R(\nu)\ \forall \nu \in I^X, \tag{8}$$

$$\mathrm{int}^\lambda_R(\nu^c) = (\mathrm{cl}^\lambda_R(\nu))^c \quad \text{and} \quad \mathrm{cl}^\lambda_R(\nu^c) = (\mathrm{int}^\lambda_R(\nu))^c\ \forall \nu \in I^X. \tag{9}$$

Definition 1.3. Let (X, R) be a fuzzy approximation space associated with $\lambda \in I^X$. Then,

(1) μ is fuzzy preopen (resp. preclosed) set iff $\mu \leq \mathrm{int}^\lambda_R(\mathrm{cl}^\lambda_R(\mu))$ (resp. $\mu \geq \mathrm{cl}^\lambda_R(\mathrm{int}^\lambda_R(\mu))$).
(2) The fuzzy preinterior of μ, denoted by $\mathrm{p\,int}^\lambda_R(\mu)$ is defined by $\mathrm{p\,int}^\lambda_R(\mu) = \bigvee\{\nu \in I^X : \mu \geq \nu,\ \nu \text{ is fuzzy preopen}\}$.
(3) The fuzzy preclosure of μ, denoted by $\mathrm{p\,cl}^\lambda_R(\mu)$ is defined by $\mathrm{p\,cl}^\lambda_R(\mu) = \bigwedge\{\nu \in I^X : \mu \leq \nu,\ \nu \text{ is fuzzy preclosed}\}$.

2. Fuzzy ideal approximation spaces

A subset $\ell \subset I^X$ is called a fuzzy ideal ([23]) on X if it satisfies the following conditions:

(1) $\overline{0} \in \ell$,
(2) If $\nu \leq \mu$ and $\mu \in \ell$, then $\nu \in \ell$ for all $\mu, \nu \in I^X$,
(3) If $\mu \in \ell$ and $\nu \in \ell$, then $(\mu \vee \nu) \in \ell$ for all $\mu, \nu \in I^X$.

If ℓ_1 and ℓ_2 are fuzzy ideals on X, we have ℓ_1 is finer than ℓ_2 (ℓ_2 is coarser than ℓ_1) if $\ell_1 \supseteq \ell_2$. The triple (X, R, ℓ) is called a fuzzy ideal approximation space. Denote the trivial fuzzy ideal ℓ° as a fuzzy ideal including only $\overline{0}$.

Definition 2.1. Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then,

(1) The local fuzzy closed set $\Phi_\lambda(\mu)(R, \ell)$ of a set $\mu \in I^X$ is defined by:

$$\Phi_\lambda(\mu)(R, \ell) = \bigwedge\{\nu \in I^X : (\mu \bar{\wedge} \nu) \in \ell,\ \ \mathrm{cl}^\lambda_R(\nu) = \nu\}. \tag{10}$$

We will write $\Phi_\lambda(\mu)$ or $\Phi_\lambda(\mu)(\ell)$ instead of $\Phi_\lambda(\mu)(R, \ell)$.

(2) The local fuzzy preclosed set $\Phi^p_\lambda(\mu)(R, \ell)$ of a set $\mu \in I^X$ is defined by:

$$\Phi^p_\lambda(\mu)(R, \ell) = \bigwedge\{\nu \in I^X : (\mu \bar{\wedge} \nu) \in \ell,\ \ \mathrm{p\,cl}^\lambda_R(\nu) = \nu\}. \tag{11}$$

We will write $\Phi^p_\lambda(\mu)$ or $\Phi^p_\lambda(\mu)(\ell)$ instead of $\Phi^p_\lambda(\mu)(R, \ell)$.

Corollary 2.1. *Let (X, R, ℓ°) be a fuzzy ideal approximation space, $\lambda \in I^X$. Then, for each $\mu \in I^X$, we have $\Phi_\lambda(\mu) = \mathrm{cl}^\lambda_R(\mu)$, $\Phi^p_\lambda(\mu) = \mathrm{p\,cl}^\lambda_R(\mu)$.*

Proposition 2.1. *Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then,*

(1) $\mu \leq \nu$ *implies* $\Phi_\lambda(\mu) \leq \Phi_\lambda(\nu)$ *and* $\Phi^p_\lambda(\mu) \leq \Phi^p_\lambda(\nu)$.
(2) *If* ℓ_1, ℓ_2 *are fuzzy ideals on* X *and* $\ell_1 \subseteq \ell_2$, *then* $\Phi_\lambda(\mu)(\ell_1) \geq \Phi_\lambda(\mu)(\ell_2)$ *and* $\Phi^p_\lambda(\mu)(\ell_1) \geq \Phi^p_\lambda(\mu)(\ell_2)$.
(3) $\Phi^p_\lambda(\mu) \leq \Phi_\lambda(\mu) = \mathrm{cl}^\lambda_R(\Phi_\lambda(\mu)) \leq \mathrm{cl}^\lambda_R(\mu)$, *and* $\Phi^p_\lambda(\mu) = \mathrm{p\,cl}^\lambda_R(\Phi^p_\lambda(\mu)) \leq \mathrm{p\,cl}^\lambda_R(\mu) \leq \mathrm{cl}^\lambda_R(\mu)$.
(4) $(\Phi_\lambda(\Phi_\lambda(\mu)) \leq \mathrm{cl}^\lambda_R(\Phi_\lambda(\mu)) = \Phi_\lambda(\mu)$,
(5) $\Phi^p_\lambda(\Phi^p_\lambda(\mu)) \leq \mathrm{pcl}^\lambda_R(\Phi^p_\lambda(\mu)) = \Phi^p_\lambda(\mu)$.
(6) $\Phi_\lambda(\mu) \vee \Phi_\lambda(\nu) \leq \Phi_\lambda(\mu \vee \nu)$ *and* $\Phi_\lambda(\mu) \wedge \Phi_\lambda(\nu) \geq \Phi_\lambda(\mu \wedge \nu)$.

Proof. Obvious. □

Definition 2.2. Let (X, R, ℓ) be a fuzzy ideal approximation space associated with
$\lambda \in I^X$. Then, for any $\mu \in I^X$, define the fuzzy operators
$\mathrm{cl}^\lambda_\Phi,\ \mathrm{p\,cl}^\lambda_\Phi,\ \mathrm{int}^\lambda_\Phi,\ \mathrm{p\,int}^\lambda_\Phi\ :\ I^X \to I^X$ as follows:

$$\mathrm{cl}^\lambda_\Phi(\mu) = \mu \vee \Phi_\lambda(\mu), \quad \mathrm{p\,cl}^\lambda_\Phi(\mu) = \mu \vee \Phi^p_\lambda(\mu)\ \forall \mu \in I^X. \tag{12}$$

$$\mathrm{int}^\lambda_\Phi(\mu) = \mu \wedge (\Phi_\lambda(\mu^c))^c, \quad \mathrm{p\,int}^\lambda_\Phi(\mu) = \mu \wedge (\Phi^p_\lambda(\mu^c))^c\ \forall \mu \in I^X. \tag{13}$$

Now, if $\ell = \ell^\circ$, then from Corollary 2.1,
(1) $\mathrm{cl}^\lambda_\Phi(\mu) = \mathrm{cl}^\lambda_R(\mu) = \Phi_\lambda(\mu)$ and $\mathrm{int}^\lambda_\Phi(\mu) = \mathrm{int}^\lambda_R(\mu) = (\Phi_\lambda(\mu^c))^c\ \ \forall \mu \in I^X$.
(2) $\mathrm{p\,cl}^\lambda_\Phi(\mu) = \mathrm{p\,cl}^\lambda_R(\mu) = \Phi^p_\lambda(\mu)$ and $\mathrm{p\,int}^\lambda_\Phi(\mu) = \mathrm{p\,int}^\lambda_R(\mu) = (\Phi^p_\lambda(\mu^c))^c\ \ \forall \mu \in I^X$.

Proposition 2.2. *Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then, for any $\mu, \nu \in I^X$, we have:*

(1) $\mathrm{int}^\lambda_R(\mu) \leq \mathrm{p\,int}^\lambda_\Phi(\mu) \leq \mathrm{int}^\lambda_\Phi(\mu) \leq \mu \leq \mathrm{p\,cl}^\lambda_\Phi(\mu) \leq \mathrm{cl}^\lambda_\Phi(\mu) \leq \mathrm{cl}^\lambda_R(\mu)$.
(2) $\mathrm{cl}^\lambda_\Phi(\mu^c) = (\mathrm{int}^\lambda_\Phi(\mu))^c$ and $\mathrm{int}^\lambda_\Phi(\mu^c) = (\mathrm{cl}^\lambda_\Phi(\mu))^c$.
(3) $\mathrm{cl}^\lambda_\Phi(\mu \vee \nu) \geq \mathrm{cl}^\lambda_\Phi(\mu) \vee \mathrm{cl}^\lambda_\Phi(\nu)$, $\mathrm{cl}^\lambda_\Phi(\mu \wedge \nu) \leq \mathrm{cl}^\lambda_\Phi(\mu) \wedge \mathrm{cl}^\lambda_\Phi(\nu)$.
(4) $\mathrm{int}^\lambda_\Phi(\mu \vee \nu) \geq \mathrm{int}^\lambda_\Phi(\mu) \vee \mathrm{int}^\lambda_\Phi(\nu)$, $\mathrm{int}^\lambda_\Phi(\mu \wedge \nu) \leq \mathrm{int}^\lambda_\Phi(\mu) \wedge \mathrm{int}^\lambda_\Phi(\nu)$.
(5) $\mathrm{cl}^\lambda_\Phi(\mathrm{cl}^\lambda_\Phi(\mu)) \geq \mathrm{cl}^\lambda_\Phi(\mu)$ and $\mathrm{int}^\lambda_\Phi(\mathrm{int}^\lambda_\Phi(\mu)) \leq \mathrm{int}^\lambda_\Phi(\mu)$.
(6) If $\mu \leq \nu$, then $\mathrm{cl}^\lambda_\Phi(\mu) \leq \mathrm{cl}^\lambda_\Phi(\nu)$, $\mathrm{int}^\lambda_\Phi(\mu) \leq \mathrm{int}^\lambda_\Phi(\nu)$.
(7) $\mathrm{pcl}^\lambda_\Phi(\mu) \leq \mathrm{pcl}^\lambda_R(\mu)$.

Proof. For (7): Suppose that $\mathrm{pcl}^\lambda_\Phi(\mu) \not\leq \mathrm{pcl}^\lambda_R(\mu)$, and if $\mathrm{pcl}^\lambda_R(\mu) = \nu$, then $\mu \leq \nu$ and ν is fuzzy preclosed set with $\mathrm{pcl}^\lambda_\Phi(\mu) \not\leq \nu$. But $\mu \leq \nu$ implies that $\mu \bar{\wedge} \nu \in \ell$, and thus $\Phi^p_\lambda(\mu) \leq \nu$ which means that $\mathrm{pcl}^\lambda_\Phi(\mu) = \mu \vee \Phi^p_\lambda(\mu) \leq \mu \wedge \nu \leq \nu$, which is a contradiction. Hence, $\mathrm{pcl}^\lambda_\Phi(\mu) \leq \mathrm{pcl}^\lambda_R(\mu)$.
(1) − (6): Clear. □

Definition 2.3. (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then,

(1) $\mu \in I^X$ is said to be fuzzy Φ-open if $\mu \leq \mathrm{int}^\lambda_R(\Phi_\lambda(\mu))$. The complement of fuzzy Φ-open is said to be fuzzy Φ-closed.
(2) $\mu \in I^X$ is called fuzzy dense in itself if $\mu \leq \Phi_\lambda(\mu)$.
(3) $\mu \in I^X$ is said to be fuzzy ideal preopen if $\mu \leq \mathrm{int}^\lambda_R(\mathrm{cl}^\lambda_\Phi(\mu))$. The complement of fuzzy ideal preopen is said to be fuzzy ideal preclosed.

Lemma 2.1. *Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then,*

(1) *If* $\mu \in I^X$ *is fuzzy Φ-closed, then* $\mu \geq \Phi_\lambda(\mathrm{int}^\lambda_R(\mu))$.
(2) *If* $\mu \in I^X$ *is fuzzy ideal preclosed, then* $\mu \geq \mathrm{cl}^\lambda_R(\mathrm{int}^\lambda_\Phi(\mu))$.

Proof. For (1): Let μ be fuzzy Φ-closed. Then, $\mu^c \leq \mathrm{int}^\lambda_R(\Phi_\lambda(\mu^c)) \leq \mathrm{int}^\lambda_R\,(\mathrm{cl}^\lambda_R\,(\mu^c)) = \mathrm{int}^\lambda_R\,((\mathrm{int}^\lambda_R(\mu))^c) = (\mathrm{cl}^\lambda_R(\mathrm{int}^\lambda_R(\mu)))^c \leq (\Phi_\lambda(\mathrm{int}^\lambda_R(\mu)))^c$. Therefore, $\Phi_\lambda(\mathrm{int}^\lambda_R(\mu)) \leq \mu$. For (2), it is easy. □

It is clear that:

$$\begin{tikzcd}
\Phi - \text{open } (\Phi - \text{closed}) \arrow[r] \arrow[dr] & \text{ideal preopen (ideal preclosed)} \arrow[d] \\
& \text{preopen (preclosed)}
\end{tikzcd}$$

Example 2.1. Let R be a fuzzy relation on a set $X = \{a, b, c, d\}$ defined as follows.

R	a	b	c	d
a	1	1	0	0
b	1	1	0	0
c	0	0	1	0.6
d	0	0	0.6	1

Assume that $\lambda = \{0, 0, 0.5, 0.5\}$ and a fuzzy ideal ℓ on X is defined as follows: $\nu \in \ell \Leftrightarrow \nu \leq \{0.5, 0.5, 1, 1\}$. Then, $\mu = \{0.3, 0.3, 1, 1\} \in \ell$ is a fuzzy preopen but it is neither fuzzy ideal preopen nor fuzzy Φ-open.

Example 2.2. Let R be a fuzzy relation on a set $X = \{a, b, c, d, e\}$ defined as follows.

R	a	b	c	d	e
a	1	1	1	0	0
b	1	1	1	0	0
c	1	1	1	0	0
d	0	0	0	1	0.2
e	0	0	0	0.2	1

Assume that $\lambda = \{1, 1, 1, 0.8, 0.6\}$ and a fuzzy ideal ℓ on X is defined by: $\nu \in \ell \Leftrightarrow \nu \leq \overline{0.6}$. Then, $\mu = \{1, 1, 1, 0, 0\} \notin \ell$ is a fuzzy ideal preopen but it is not fuzzy Φ-open.

Theorem 2.1. *Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then, the following are equivalent.*

(1) *$\mu \in I^X$ is fuzzy Φ-open.*
(2) *$\mu \in I^X$ is fuzzy ideal preopen and fuzzy ideal dense in itself.*

Proof. (1) $\Rightarrow$ (2): It is clear that every fuzzy Φ-open set is fuzzy ideal preopen. On the other hand $\mu \leq \mathrm{int}^{\lambda}_{R}(\Phi_{\lambda}(\mu)) \leq \Phi_{\lambda}(\mu)$, which means μ is fuzzy ideal dense in itself.

(2) $\Rightarrow$ (1): By assumption, $\mu \leq \mathrm{int}^{\lambda}_{R}(\mathrm{cl}^{\lambda}_{\Phi}(\mu)) = \mathrm{int}^{\lambda}_{R}(\mu \vee \Phi_{\lambda}(\mu)) = \mathrm{int}^{\lambda}_{R}(\Phi_{\lambda}(\mu))$, and hence μ is fuzzy Φ-open. $\square$

The following example shows that fuzzy ideal preopen and fuzzy ideal dense in itself are independent concepts.

Example 2.3.

(1) In Example 2.2, we get that: For $\mu = \{1, 1, 1, 0, 0\}$, we have μ is a fuzzy ideal preopen set but not fuzzy ideal dense in itself.
(2) Let R be a fuzzy relation on a set $X = \{a, b, c, d\}$ defined as follows.

R	a	b	c	d
a	1	1	0	0
b	1	1	0	0
c	0	0	1	0.8
d	0	0	0.8	1

Assume that $\lambda = \{1, 1, 0.2, 0\}$ and a fuzzy ideal ℓ on X is defined as follows: $\nu \in \ell \Leftrightarrow \nu \leq \overline{0.2}$. Then, $\mu = \{0.6, 0.5, 0.1, 0.1\}$ is a fuzzy ideal dense in itself. But it is not fuzzy ideal preopen set.

3. Separation axioms in fuzzy ideal approximation spaces

Definition 3.1. Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then,

(1) A fuzzy ideal approximation space (X, R, ℓ) (resp. a fuzzy approximation space (X, R)) is called a fuzzy ideal-$(t, s)T_0$ (resp. $(t, s)T_0$) if for every $x \neq y \in X$, there exists $\mu \in I^X$, $t \in I_0$ with $\mathrm{int}^{\lambda}_{\Phi}(\mu)(x) \geq t$ (resp. $\mathrm{int}^{\lambda}_{R}(\mu)(x) \geq t$) such that $\mu(y) < t$ or there exists $\nu \in I^X$, $s \in I_0$ with $\mathrm{int}^{\lambda}_{\Phi}(\nu)(y) \geq s$ (resp. $\mathrm{int}^{\lambda}_{R}(\nu)(y) \geq s$) such that $\nu(x) < s$.
(2) A fuzzy ideal approximation space (X, R, ℓ) (resp. a fuzzy approximation space (X, R)) is called a fuzzy ideal-$(t, s)T_1$ (resp. $(t, s)T_1$) if for every $x \neq y \in X$, there exist $\mu, \nu \in I^X$; $t, s \in I_0$ with $\mathrm{int}^{\lambda}_{\Phi}(\mu)(x) \geq t$ and $\mathrm{int}^{\lambda}_{\Phi}(\nu)(y) \geq s$ (resp. $\mathrm{int}^{\lambda}_{R}(\mu)(x) \geq t$ and $\mathrm{int}^{\lambda}_{R}(\nu)(y) \geq s$) such that $\mu(y) < t$ and $\nu(x) < s$.
(3) A fuzzy ideal approximation space (X, R, ℓ) (resp. a fuzzy approximation space (X, R)) is called a fuzzy ideal-$(t, s)T_2$ (resp. $(t, s)T_2$) if for every $x \neq y \in X$, there exist $\mu, \nu \in I^X$; $t, s \in I_0$ with $\mathrm{int}^{\lambda}_{\Phi}(\mu)(x) \geq t$ and $\mathrm{int}^{\lambda}_{\Phi}(\nu)(y) \geq s$ (resp. $\mathrm{int}^{\lambda}_{R}(\mu)(x) \geq t$ and $\mathrm{int}^{\lambda}_{R}(\nu)(y) \geq s$) such that $\sup(\mu \wedge \nu) < (t \wedge s)$.

Remark 3.1 From (1) in Proposition 2.2, we have $\mathrm{int}^{\lambda}_{\Phi}(\mu) \geq \mathrm{int}^{\lambda}_{R}(\mu)\ \forall \mu \in I^X$. Denote for fuzzy ideal approximation $(t, s)T_i$ separation axioms by $(t, s)FI - T_i$, $i = 0, 1, 2$, that is,

$$\begin{tikzcd}
(t,s)T_2 \arrow[r] \arrow[d] & (t,s)T_1 \arrow[r] \arrow[d] & (t,s)T_0 \arrow[d] \\
(t,s)FI - T_2 \arrow[r] & (t,s)FI - T_1 \arrow[r] & (t,s)FI - T_0
\end{tikzcd}$$

Consider a fuzzy ideal approximation space (X, R, ℓ) associated with $\lambda \in I^X$ and $\ell = \{\overline{0}\}$. Then, the fuzzy ideal separation axioms $(t, s)FI - T_i$ are identical to the fuzzy separation axioms $(t, s)T_i$ of the fuzzy approximation space (X, R), $i = 0, 1, 2$.

Example 3.1. Let $\lambda = \{1, 0.8, 0\}$, $t = s = 0.5$ and R be a fuzzy relation on a set $X = \{a, b, c\}$ as shown in the matrix:

R	a	b	c
a	1	0.3	0
b	0.3	1	0
c	0	0	1

Then, we get that: $\lambda_R = \{0.7, 0.8, 0\}$, $\lambda^R = \{1, 0.8, 0\}$, $\lambda^c_R = \{0.3, 0.2, 1\}$.

Now, for the case $a \neq b$, there exists $\mu = \{0.8, 0, 0.4\}$, and then $\mu_R = \{0.7, 0, 0.4\}$, which means $\mathrm{int}^\lambda_R(\mu) = \{0.7, 0, 0\}$, and thus $\mathrm{int}^\lambda_R(\mu)(a) \geq 0.5$, $\mu(b) < 0.5$. Also, we can find $\nu = \{0, 0.6, 0.1\}$, and then $\nu_R = \{0, 0.6, 0.1\}$, which means $\mathrm{int}^\lambda_R(\nu) = \{0, 0.6, 0\}$, and thus $\mathrm{int}^\lambda_R(\nu)(b) \geq 0.5$, $\nu(a) < 0.5$.

For the cases $a \neq c$ and $b \neq c$, we can find $\eta \in I^X$ with $\mathrm{int}^\lambda_R(\eta)(a) \geq 0.5$ or $\mathrm{int}^\lambda_R(\eta)(b) \geq 0.5$ such that $\eta(c) < 0.5$, while we can not find $\eta \in I^X$ with $\mathrm{int}^\lambda_R(\eta)(c) \geq 0.5$. Hence, (X, R) is a fuzzy approximation $(0.5, 0.5)T_0$-space associated with λ. (X, R) could not be a fuzzy approximation $(0.5, 0.5)T_1$-space or $(0.5, 0.5)T_2$-space. Now, any fuzzy set ω will satisfy $\mathrm{cl}^\lambda_R(\omega) = \omega \iff \omega \geq \{0.3, 0.3, 1\}$ according to the fuzzy cosets of R and the set λ^c_R.

Define a fuzzy ideal ℓ on X so that $\eta \in \ell \iff \eta \leq \overline{0.7}$. Then, we can find three fuzzy sets $\eta = \{0.8, 0, 0\}$, $\xi = \{0, 0.8, 0\}$ and $\zeta = \{0, 0, 0.8\}$ for which $\Phi_\lambda(\eta^c) = \Phi_\lambda(\xi^c) = \Phi_\lambda(\zeta^c) = \{0.3, 0.3, 1\}$, and then $\mathrm{int}^\lambda_\Phi(\eta) = \eta \wedge (\Phi_\lambda(\eta^c))^c = \{0.7, 0, 0\}$, $\mathrm{int}^\lambda_\Phi(\xi) = \xi \wedge (\Phi_\lambda(\xi^c))^c = \{0, 0.7, 0\}$ and $\mathrm{int}^\lambda_\Phi(\zeta) = \zeta \wedge (\Phi_\lambda(\zeta^c))^c = \{0, 0, 0.7\}$. Thus, for any $x \neq y$, we have two fuzzy sets $\rho, \sigma \in \{\eta, \xi, \zeta\}$ so that $\mathrm{int}^\lambda_\Phi(\sigma)(x) \geq 0.5$, $\sigma(y) < 0.5$ and $\mathrm{int}^\lambda_\Phi(\rho)(y) \geq 0.5$, $\rho(x) < 0.5$. Hence, for any choice for ρ, σ, we have $\sup(\rho \wedge \sigma) = 0 < 0.5$, and therefore (X, R, ℓ) is a fuzzy ideal approximation $(0.5, 0.5)T_i$-space, $i = 0, 1, 2$ while (X, R) is even not fuzzy approximation $(0.5, 0.5)T_1$-space.

The following example is given to show that there is a fuzzy ideal approximation $(t, s)T_0$-space but not fuzzy approximation $(t, s)T_0$-space.

Example 3.2. Let $\lambda = \{0.6, 0, 0\}$, $t = s = 0.4$ and R be a fuzzy relation on a set $X = \{a, b, c\}$ as shown in the matrix:

R	a	b	c
a	1	0	0
b	0	1	0
c	0	0	1

Then, we get that: $\lambda_R = \{0.6, 0, 0\}$, $\lambda^c_R = \{0.4, 1, 1\}$. Now, for the case $b \neq c$, we can not find $\eta \in I^X$ with $\mathrm{int}^\lambda_R(\eta)(b) \geq 0.4$ or $\mathrm{int}^\lambda_R(\eta)(c) \geq 0.4$. Hence, (X, R) is not fuzzy approximation $(0.4, 0.4)T_0$-space associated with λ. Consequently, (X, R) could not be a fuzzy approximation $(0.4, 0.4)T_1$-space or $(0.4, 0.4)T_2$-space.

Define a fuzzy ideal ℓ on X so that $\eta \in \ell \iff \eta \leq \{0.6, 1, 1\}$. Then, there exist $\mu = \{0.4, 0.4, 0\}$ and $\nu = \{0.4, 0, 0.4\}$ for which $\Phi_\lambda(\mu^c) = \overline{0}$ and $\Phi_\lambda(\nu^c) = \overline{0}$, which implies that $\mathrm{int}^\lambda_\Phi(\mu) = \mu = \{0.4, 0.4, 0\}$ and $\mathrm{int}^\lambda_\Phi(\nu) = \nu = \{0.4, 0, 0.4\}$, and thus $\mathrm{int}^\lambda_\Phi(\mu)(a) \geq 0.4$, $\mu(c) < 0.4$, $\mathrm{int}^\lambda_\Phi(\mu)(b) \geq 0.4$, $\mu(c) < 0.4$ and $\mathrm{int}^\lambda_\Phi(\nu)(a) \geq 0.4$, $\nu(b) < 0.4$. That is, (X, R, ℓ) is a fuzzy ideal approximation $(0.4, 0.4)T_0$-space but (X, R) is not fuzzy approximation $(0.4, 0.4)T_0$-space.

If (X, R) and (Y, R^*) are fuzzy approximation spaces associated with $\lambda \in I^X$ and $\mu \in I^Y$, respectively, then a mapping $f : (X, R) \to (Y, R^*)$ is said to be fuzzy approximation continuous (FAC) if $\mathrm{int}^\lambda_R(f^{-1}(\eta)) \geq f^{-1}(\mathrm{int}^\mu_{R^*}(\eta))\ \forall \eta \in I^Y$. It is equivalent to $\mathrm{cl}^\lambda_R(f^{-1}(\eta)) \leq f^{-1}(\mathrm{cl}^\mu_{R^*}(\eta))\ \forall \eta \in I^Y$.

Now, with respect to $\lambda \in I^X$ and $\mu \in I^Y$, if ℓ, ℓ^* are fuzzy ideals on X, Y, respectively, then a mapping $f : (X, R, \ell) \to (Y, R^*)$ is called fuzzy ideal approximation continuous (FIAC) provided that $\mathrm{int}^\lambda_\Phi(f^{-1}(\eta)) \geq f^{-1}(\mathrm{int}^\mu_{R^*}(\eta))\ \forall \eta \in I^Y$. It is easily shown that it is equivalent to $\mathrm{cl}^\lambda_\Phi(f^{-1}(\eta)) \leq f^{-1}(\mathrm{cl}^\mu_{R^*}(\eta))\ \forall \eta \in I^Y$. Also, let us call $f : (X, R) \to (Y, R^*)$ a fuzzy approximation open (FAO) provided that $\mathrm{int}^\mu_{R^*}(f(\xi)) \geq f(\mathrm{int}^\lambda_R(\xi))\ \forall \xi \in I^X$, $f : (X, R) \to (Y, R^*, \ell^*)$ a fuzzy ideal approximation open (FIAO) provided that $\mathrm{int}^\mu_\Phi(f(\xi)) \geq f(\mathrm{int}^\lambda_R(\xi))\ \forall \xi \in I^X$.

Clearly, every (FAC) (resp. (FAO)) mapping will be (FIAC) (resp. (FIAO)) mapping as well (from (1) in Proposition 2.2).

Theorem 3.1. *Let (X, R), (Y, R^*) be fuzzy approximation spaces associated with $\lambda \in I^X$, $\mu \in I^Y$, respectively, ℓ a fuzzy ideal on X and $f : (X, R) \to (Y, R^*)$ is an injective (FAC) mapping with $f(\lambda) = \mu$. Then, (X, R, ℓ) is a fuzzy ideal approximation $(t, s)T_i$-space if (Y, R^*) is a fuzzy approximation $(t, s)T_i$-space, $i = 0, 1, 2$.*

Proof. Since $x \neq y$ in X implies that $f(x) \neq f(y)$ in Y, and from Y is a fuzzy approximation $(t, s)T_2$-space, then there exist $\eta, \zeta \in I^Y$ with $t \leq \mathrm{int}^\mu_{R^*}(\eta)(f(x))$, $s \leq \mathrm{int}^\mu_{R^*}(\zeta)(f(y))$ such that $\sup(\eta \wedge \zeta) < (t \wedge s)$, that is, $t \leq f^{-1}(\mathrm{int}^\mu_{R^*}(\eta))(x)$, $s \leq f^{-1}(\mathrm{int}^\mu_{R^*}(\zeta))(y)$, and then $t \leq f^{-1}(\mathrm{int}^\mu_\Phi(\eta))(x)$, $s \leq f^{-1}(\mathrm{int}^\mu_\Phi(\zeta))(y)$. Since f is (FAC), then $t \leq \mathrm{int}^\lambda_R(f^{-1}(\eta))(x)$, $s \leq \mathrm{int}^\lambda_R(f^{-1}(\zeta))(y)$, and then $t \leq \mathrm{int}^\lambda_\Phi(f^{-1}(\eta))(x)$, $s \leq$

$\text{int}_{\Phi}^{\lambda}(f^{-1}(\zeta))(y)$. That is, there exist $\rho = f^{-1}(\eta)$, $\omega = f^{-1}(\zeta)$ with $t \le \text{int}_{\Phi}^{\lambda}(\rho)(x)$, $s \le \text{int}_{\Phi}^{\lambda}(\omega)(y)$ and $\sup(\rho \wedge \omega) < (t \wedge s)$. Hence, (X, R, ℓ) is a fuzzy ideal approximation $(t, s)T_2$-space. Other cases are similar. $\square$

Theorem 3.2. *Let (X, R), (Y, R^*) be fuzzy approximation spaces associated with $\lambda \in I^X$, $\mu \in I^Y$, respectively, ℓ^* a fuzzy ideal on Y and $f : (X, R) \to (Y, R^*)$ is a surjective (FAO) mapping with $f^{-1}(\mu) = \lambda$. Then, (Y, R^*, ℓ^*) is a fuzzy ideal $(t, s)T_i$-space if (X, R) is a fuzzy approximation $(t, s)T_i$-space, $i = 0, 1, 2$.*

Proof. Since f is surjective, then $p \neq q$ in Y implies that $f^{-1}(p) \neq f^{-1}(q)$ in X, and from (X, R) is a fuzzy approximation $(t, s)T_2$-space, then there exist $\rho, \omega \in I^X$ with $t \le \text{int}_{R}^{\lambda}(\rho)(f^{-1}(p))$, $s \le \text{int}_{R}^{\lambda}(\omega)(f^{-1}(q))$ such that $\sup(\rho \wedge \omega) < (t \wedge s)$, and also from f is surjective, then $f(\text{int}_{R}^{\lambda}(\rho))(p) = \text{int}_{R}^{\lambda}(\rho)(f^{-1}(p))$ and $f(\text{int}_{R}^{\lambda}(\omega))(q) = \text{int}_{R}^{\lambda}(\omega)(f^{-1}(q))$, and thus $t \le f(\text{int}_{R}^{\lambda}(\rho))(p)$, $s \le f(\text{int}_{R}^{\lambda}(\omega))(q)$. From f is (FAO), then, $t \le \text{int}_{R^*}^{\mu}(f(\rho))(p)$, $s \le \text{int}_{R^*}^{\mu}(f(\omega))(q)$, and thus $t \le \text{int}_{\Phi}^{\mu}(f(\rho))(p)$, $s \le \text{int}_{\Phi}^{\mu}(f(\omega))(q)$. That is, there exist $\eta = f(\rho)$, $\zeta = f(\omega)$ with $t \le \text{int}_{\Phi}^{\mu}(\eta)(p)$, $s \le \text{int}_{\Phi}^{\mu}(\zeta)(q)$ and $\sup(\eta \wedge \zeta) < (t \wedge s)$. Hence, (Y, R^*, ℓ^*) is a fuzzy ideal approximation $(t, s)T_2$-space. The other cases for $(t, s)T_0$-spaces and $(t, s)T_1$-spaces are similar. $\square$

4. Connected fuzzy ideal approximation spaces

Definition 4.1. Let (X, R) be a fuzzy approximation space associated with $\lambda \in I^X$. Then,

(1) The fuzzy sets $\mu, \nu \in I^X$ are called fuzzy approximation preseparated (resp. separated) sets if $\text{p}\,\text{cl}_{R}^{\lambda}(\mu) \wedge \nu = \mu \wedge \text{p}\,\text{cl}_{R}^{\lambda}(\nu) = \overline{0}$ (resp. $\text{cl}_{R}^{\lambda}(\mu) \wedge \nu = \mu \wedge \text{cl}_{R}^{\lambda}(\nu) = \overline{0}$).

(2) A fuzzy set $\eta \in I^X$ is called fuzzy approximation predisconnected (resp. disconnected) set if there exist fuzzy approximation preseparated (resp. separated) sets $\mu, \nu \in I^X$, such that $\mu \vee \nu = \eta$. A fuzzy set η is called fuzzy approximation preconnected (resp. connected) if it is not fuzzy approximation predisconnected (resp. disconnected).

(3) (X, R) is called fuzzy approximation predisconnected (resp. disconnected) space if there exist fuzzy approximation preseparated (resp. separated) sets $\mu, \nu \in I^X$, such that $\mu \vee \nu = \overline{1}$. A fuzzy approximation space(X, R) is called fuzzy approximation preconnected (resp. connected) space if it is not fuzzy approximation predisconnected (resp. disconnected) space.

Definition 4.2. Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then,

(1) the fuzzy sets $\mu, \nu \in I^X$ are called fuzzy ideal approximation preseparated (resp. separated) sets if $\text{p}\,\text{cl}_{\Phi}^{\lambda}(\mu) \wedge \nu = \mu \wedge \text{p}\,\text{cl}_{\Phi}^{\lambda}(\nu) = \overline{0}$ (resp. $\text{cl}_{\Phi}^{\lambda}(\mu) \wedge \nu = \mu \wedge \text{cl}_{\Phi}^{\lambda}(\nu) = \overline{0}$).

(2) A fuzzy set $\eta \in I^X$ is called fuzzy ideal approximation predisconnected (resp. disconnected) set if there exist fuzzy ideal approximation preseparated (resp. separated) sets $\mu, \nu \in I^X$, such that $\mu \vee \nu = \eta$. A fuzzy set η is called fuzzy ideal approximation preconnected (resp. connected) if it is not fuzzy ideal approximation predisconnected (resp. disconnected).

(3) (X, R, ℓ) is called fuzzy ideal approximation predisconnected (resp. disconnected) space if there exist fuzzy ideal approximation preseparated (resp. separated) sets $\mu, \nu \in I^X$, such that $\mu \vee \nu = \overline{1}$. A fuzzy ideal approximation space(X, R, ℓ) is called fuzzy ideal approximation preconnected (resp. connected) space if it is not fuzzy ideal approximation predisconnected (resp. disconnected) space.

Remark 4.1. We have the following implications.

$$\begin{tikzcd}
\textit{fuzzy separated} \arrow[r] \arrow[d] & \textit{fuzzy ideal separated} \arrow[d] \\
\textit{fuzzy preseparated} \arrow[r] & \textit{fuzzy ideal preseparated}
\end{tikzcd}$$

and hence,

$$\begin{tikzcd}
\textit{fuzzy ideal preconnected} \arrow[r] \arrow[d] & \textit{fuzzy preconnected} \arrow[d] \\
\textit{fuzzy ideal connected} \arrow[r] & \textit{fuzzy connected}
\end{tikzcd}$$

Example 4.1. Let $X = \{a, b, c, d, e\}$ and R a fuzzy relation on X defined by

R	a	b	c	d	e
a	1	1	0.2	0	0
b	1	1	0.2	0	0
c	0.2	0.2	1	0	0
d	0	0	0	1	0
e	0	0	0	0	1

Suppose that $\lambda = \{0, 0, 0.4, 0.8, 0\}$. Then, $\lambda_R = \{0, 0, 0.4, 0.8, 0\}$, and $(\lambda_R)^c = \{1, 1, 0.6, 0.2, 1\}$. Now, for $\mu = \{0.6, 0, 0, 0, 0\}$, $\nu = \{0, 0.6, 0, 0, 0\}$. Then, $\mu^R = \{0.6, 1, 0.2, 0, 0\}$, $\nu^R = \{1, 0.6, 0.2, 0, 0\}$, and thus $\mathrm{cl}^{\lambda}_R(\mu) = \{1, 1, 0.6, 0.2, 1\}$ and $\mathrm{cl}^{\lambda}_R(\nu) = \{1, 1, 0.6, 0.2, 1\}$. Moreover, $\mu_R = \overline{0}$, $\nu_R = \overline{0}$, and thus $\mathrm{int}^{\lambda}_R(\mu) = \overline{0}$ and $\mathrm{int}^{\lambda}_R(\nu) = \overline{0}$. Hence,

(1) μ, ν are fuzzy approximation preseparated sets but not fuzzy approximation separated sets.
(2) Consider a fuzzy ideal $\mathcal{I}$ defined on X so that $\eta \in \mathcal{I}\ \forall \eta \le \overline{0.6}$. Then, $\mu \in \mathcal{I},\ \nu \in \mathcal{I}$, which means that $\Phi_\lambda(\mu) = \overline{0}$ and $\Phi_\lambda(\nu) = \overline{0}$, and then $\mathrm{cl}^{\lambda}_{\Phi}(\mu) = \mu$ and $\mathrm{cl}^{\lambda}_{\Phi}(\nu) = \nu$. Thus, $\mathrm{cl}^{\lambda}_{\Phi}(\mu) \wedge \nu = \overline{0}$ and $\mathrm{cl}^{\lambda}_{\Phi}(\nu) \wedge \mu = \overline{0}$, and then μ, ν are fuzzy ideal approximation separated sets but not fuzzy approximation separated sets.
(3) Consider a fuzzy ideal $\mathcal{I}$ defined on X so that $\eta \in \mathcal{I} \forall \eta \le \overline{0.3}$. Then, $\mu \notin \mathcal{I},\ \nu \notin \mathcal{I}$, which implies that $\Phi_\lambda(\mu) = \Phi_\lambda(\nu) = \{1, 1, 0.6, 0.2, 1\}$, and then $\mathrm{cl}^{\lambda}_{\Phi}(\mu) = \mathrm{cl}^{\lambda}_{\Phi}(\nu) = \{1, 1, 0.6, 0.2, 1\}$. Thus, μ, ν are not fuzzy ideal approximation separated sets.
But, μ, ν are fuzzy approximation preclosed sets, then $\Phi^p_\lambda(\mu) = \mu$ and $\Phi^p_\lambda(\nu) = \nu$, and then $\mathrm{p\,cl}^{\lambda}_{\Phi}(\mu) = \mu$ and $\mathrm{p\,cl}^{\lambda}_{\Phi}(\nu) = \nu$. Hence, μ, ν are fuzzy ideal approximation preseparated sets but not fuzzy ideal approximation separated sets.
(4) Here, $\eta = \{0.6, 0, 0.6, 0, 0\}$, $\xi = \{0, 0.6, 0, 0.6, 0\}$ are not fuzzy approximation preseparated, where $\mathrm{p\,cl}^{\lambda}_R(\eta) = \{1, 1, 0.6, 0.2, 1\}$ and $\mathrm{p\,cl}^{\lambda}_R(\xi) = \{1, 1, 0.6, 0.6, 1\}$ from that $\eta_R = \{0, 0, 0.6, 0, 0\}$ and $\xi_R = \{0, 0, 0, 0.6, 0\}$, $\mathrm{int}^{\lambda}_R(\eta) = \{0, 0, 0.4, 0, 0\}$ and $\mathrm{int}^{\lambda}_R(\xi) = \{0, 0, 0, 0.6, 0\}$. While, η, ξ are fuzzy ideal approximation preseparated sets whenever $\mathcal{I}$ is a fuzzy ideal defined on X so that $\zeta \in \mathcal{I}\ \forall \zeta \le \overline{0.6}$. That is, $\Phi^p_\lambda(\eta) = \overline{0}$ and $\Phi^p_\lambda(\xi) = \overline{0}$, and then $\mathrm{p\,cl}^{\lambda}_{\Phi}(\eta) = \eta$ and $\mathrm{p\,cl}^{\lambda}_{\Phi}(\xi) = \xi$, and thus $\mathrm{p\,cl}^{\lambda}_{\Phi}(\eta) \wedge \xi = \overline{0}$ and $\mathrm{p\,cl}^{\lambda}_{\Phi}(\xi) \wedge \eta = \overline{0}$.

Proposition 4.1. *Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then, the following are equivalent.*

(1) *(X, R, ℓ) is fuzzy ideal approximation preconnected.*
(2) $\mu \wedge \nu = \overline{0}$, $\mathrm{p\,int}^{\lambda}_{\Phi}(\mu) = \mu$, $\mathrm{p\,int}^{\lambda}_{\Phi}(\nu) = \nu$ and $\mu \vee \nu = \overline{1}$ imply $\mu = \overline{0}$ or $\nu = \overline{0}$.
(3) $\mu \wedge \nu = \overline{0}$, $\mathrm{p\,cl}^{\lambda}_{\Phi}(\mu) = \mu$, $\mathrm{p\,cl}^{\lambda}_{\Phi}(\nu) = \nu$ and $\mu \vee \nu = \overline{1}$ imply $\mu = \overline{0}$ or $\nu = \overline{0}$.

Proof. (1) $\Rightarrow$ (2): Let $\mu, \nu \in I^X$ with $\mathrm{p\,int}^{\lambda}_{\Phi}(\mu) = \mu$, $\mathrm{p\,int}^{\lambda}_{\Phi}(\nu) = \nu$ such that $\mu \wedge \nu = \overline{0}$ and $\mu \vee \nu = \overline{1}$. Then,
$\mathrm{p\,cl}^{\lambda}_{\Phi}(\mu) = \mathrm{p\,cl}^{\lambda}_{\Phi}(\nu^c) = (\mathrm{p\,int}^{\lambda}_{\Phi}(\nu))^c = \nu^c = \mu$, $\mathrm{p\,cl}^{\lambda}_{\Phi}(\nu) = \mathrm{p\,cl}^{\lambda}_{\Phi}(\mu^c) = (\mathrm{p\,int}^{\lambda}_{\Phi}(\mu))^c = \mu^c = \nu$. Hence, $\mathrm{p\,cl}^{\lambda}_{\Phi}(\mu) \wedge \nu = \mu \wedge \mathrm{p\,cl}^{\lambda}_{\Phi}(\nu) = \mu \wedge \nu = \overline{0}$. That is, μ, ν are fuzzy ideal approximation preseparated sets so that $\mu \vee \nu = \overline{1}$. But (X, R, ℓ) is fuzzy ideal approximation preconnected implies that $\mu = \overline{0}$ or $\nu = \overline{0}$.

(2) $\Rightarrow$ (3): , (3) $\Rightarrow$ (1): Clear. □

Proposition 4.2. *Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then, for $\mu \in I^X$, the following are equivalent.*

(1) *μ is fuzzy ideal approximation preconnected set.*
(2) *If ν, ρ are fuzzy ideal approximation preseparated sets with $\mu \le (\nu \vee \rho)$, then* $\mu \wedge \nu = \overline{0}$ *or* $\mu \wedge \rho = \overline{0}$.
(3) *If ν, ρ are fuzzy ideal approximation preseparated sets with $\mu \le (\nu \vee \rho)$, then* $\mu \le \nu$ *or* $\mu \le \rho$.

Proof.

(1) $\Rightarrow$ (2): Let ν, ρ be fuzzy ideal approximation preseparated sets with $\mu \le (\nu \vee \rho)$. That is, $\mathrm{pcl}^{\lambda}_{\Phi}(\nu) \wedge \rho = \mathrm{pcl}^{\lambda}_{\Phi}(\rho) \wedge \nu = \overline{0}$ so that $\mu \le (\nu \vee \rho)$. Since

$$\mathrm{pcl}^{\lambda}_{\Phi}(\mu \wedge \nu) \wedge (\mu \wedge \rho) = \mathrm{pcl}^{\lambda}_{\Phi}(\mu) \wedge \mathrm{pcl}^{\lambda}_{\Phi}(\nu) \wedge (\mu \wedge \rho)$$

$$= \mathrm{pcl}^{\lambda}_{\Phi}(\mu) \wedge \mu \wedge \mathrm{pcl}^{\lambda}_{\Phi}(\nu) \wedge \rho = \mu \wedge \overline{0} = \overline{0}.$$

$$\mathrm{pcl}^{\lambda}_{\Phi}(\mu \wedge \rho) \wedge (\mu \wedge \nu) = \mathrm{pcl}^{\lambda}_{\Phi}(\mu) \wedge \mathrm{pcl}^{\lambda}_{\Phi}(\rho) \wedge (\mu \wedge \nu)$$

$$= \mathrm{pcl}^{\lambda}_{\Phi}(\mu) \wedge \mu \wedge \mathrm{pcl}^{\lambda}_{\Phi}(\rho) \wedge \nu = \mu \wedge \overline{0} = \overline{0}.$$

Then, $(\mu \wedge \nu)$ and $(\mu \wedge \rho)$ are fuzzy ideal approximation preseparated sets with $\mu = (\mu \wedge \nu) \vee (\mu \wedge \rho)$. But μ is fuzzy ideal approximation preconnected means that $\mu \wedge \nu = \overline{0}$ or $\mu \wedge \rho = \overline{0}$.

(2) $\Rightarrow$ (3): If $\mu \wedge \nu = \overline{0}$, $\mu \leq (\nu \vee \rho)$ means that $\mu = \mu \wedge (\nu \vee \rho) = (\mu \wedge \nu) \vee (\mu \wedge \rho) = \mu \wedge \rho$, and thus $\mu \leq \rho$. Also, if $\mu \wedge \rho = \overline{0}$, then $\mu \leq \nu$.

(3) $\Rightarrow$ (1): Let ν, ρ be fuzzy ideal approximation preseparated sets so that $\mu = \nu \vee \rho$. Then, from (3), $\mu \leq \nu$ or $\mu \leq \rho$. If $\mu \leq \nu$, then

$$\rho = (\nu \vee \rho) \wedge \rho = \mu \wedge \rho \leq \nu \wedge \rho \leq \mathrm{pcl}_{\Phi}^{\lambda}(\nu) \wedge \rho = \overline{0}.$$

Also, if $\mu \leq \rho$, then $\nu = (\nu \vee \rho) \wedge \nu = \mu \wedge \nu \leq \rho \wedge \nu \leq \mathrm{pcl}_{\Phi}^{\lambda}(\rho) \wedge \nu = \overline{0}$.
Hence, μ is fuzzy ideal approximation preconnected set. □

Corollary 4.1. *Let (X, R) be a fuzzy approximation space associated with $\lambda \in I^X$. Then, for $\mu \in I^X$, the following are equivalent.*

(1) *μ is fuzzy approximation preconnected set.*
(2) *If ν, ρ are fuzzy approximation preseparated sets with $\mu \leq (\nu \vee \rho)$, then $\mu \wedge \nu = \overline{0}$ or $\mu \wedge \rho = \overline{0}$.*
(3) *If ν, ρ are fuzzy approximation preseparated sets with $\mu \leq (\nu \vee \rho)$, then $\mu \leq \nu$ or $\mu \leq \rho$.*

Theorem 4.1. *Let $(X, R), (Y, R^*)$ be fuzzy approximation spaces associated with $\lambda \in I^X$, $\mu \in I^Y$, respectively, ℓ a fuzzy ideal on X, and $f : (X, R, \ell) \to (Y, R^*)$ is a fuzzy mapping such that $\mathrm{p\,cl}_{\Phi}^{\lambda}(f^{-1}(\nu)) \leq f^{-1}(\mathrm{p\,cl}_{R^*}^{\mu}(\nu))$ $\forall \nu \in I^Y$. Then, $f(\eta) \in I^Y$ is a fuzzy approximation preconnected set if η is a fuzzy ideal approximation preconnected in X.*

Proof. Let $\nu, \rho \in I^Y$ be fuzzy approximation preseparated sets with $f(\eta) = \nu \vee \rho$. That is, $\mathrm{p\,cl}_{R^*}^{\mu}(\nu) \wedge \rho = \mathrm{p\,cl}_{R^*}^{\mu}(\rho) \wedge \nu = \overline{0}$. Then, $\eta \leq (f^{-1}(\nu) \vee f^{-1}(\rho))$, and from the condition of f, we get that

$$\mathrm{p\,cl}_{\Phi}^{\lambda}(f^{-1}(\nu)) \wedge f^{-1}(\rho) \leq f^{-1}(\mathrm{p\,cl}_{R^*}^{\mu}(\nu)) \wedge f^{-1}(\rho)$$
$$= f^{-1}(\mathrm{p\,cl}_{R^*}^{\mu}(\nu) \wedge \rho) = f^{-1}(\overline{0}) = \overline{0},$$

and in similar way, we have

$$\mathrm{p\,cl}_{\Phi}^{\lambda}(f^{-1}(\rho)) \wedge f^{-1}(\nu) \leq f^{-1}(\mathrm{p\,cl}_{R^*}^{\mu}(\rho)) \wedge f^{-1}(\nu)$$
$$= f^{-1}(\mathrm{cl}_{R^*}^{\mu}(\rho) \wedge \nu) = f^{-1}(\overline{0}) = \overline{0}.$$

Hence, $f^{-1}(\nu)$ and $f^{-1}(\rho)$ are fuzzy ideal approximation preseparated sets in X so that $\eta \leq (f^{-1}(\nu) \vee f^{-1}(\rho))$. Since η is fuzzy ideal approximation preconnected, then from (3) in Proposition 4.2, we get that $\eta \leq f^{-1}(\nu)$ or $\eta \leq f^{-1}(\rho)$, which means that $f(\eta) \leq \nu$ or $f(\eta) \leq \rho$. Thus, from Corollary 4.1, $f(\eta)$ is fuzzy approximation preconnected in Y. □

5. Compactness in fuzzy ideal approximation spaces

This section is devoted to the notion of fuzzy ideal approximation compact spaces.

Definition 5.1. Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then, X is said to be fuzzy regular (resp. fuzzy ideal regular) space if for each $\eta \in I^X$ with $\mathrm{int}_R^{\lambda}(\eta) = \eta$,

$$\eta = \bigvee_{j \in J} \{\eta_j \ : \ \mathrm{int}_R^{\lambda}(\eta_j) = \eta_j, \ \mathrm{cl}_R^{\lambda}(\eta_j) \leq \eta\}.$$

$$(\text{ resp. } \eta = \bigvee_{j \in J} \{\eta_j \ : \ \mathrm{int}_R^{\lambda}(\eta_j) = \eta_j, \ \mathrm{cl}_{\Phi}^{\lambda}(\eta_j) \leq \eta\}).$$

It is clear that every fuzzy regular space is a fuzzy ideal regular space. If $\ell = \{\overline{0}\}$, then the concepts of fuzzy regular and fuzzy ideal regular are identical.

Definition 5.2. Let (X, R, ℓ) be a fuzzy ideal approximation space associated with $\lambda \in I^X$. Then,

(1) μ is said to be fuzzy approximation compact (resp. fuzzy ideal approximation compact) if for any family $\{\mu_j \in I^X : \ \mathrm{int}_R^{\lambda}(\mu_j) = \mu_j, j \in J\}$ with $\mu \leq \bigvee_{j \in J} \mu_j$, there exists a finite subset J_0 of J such that $\mu \leq \bigvee_{j \in J_0} \mu_j$ (resp. $\mu \,\overline{\wedge}\, (\bigvee_{j \in J_0} \mu_j) \in \ell$).
(2) μ is said to be fuzzy almost approximation compact (resp. fuzzy almost ideal approximation compact) if for any family $\{\mu_j \in I^X :$ $\mathrm{int}_R^{\lambda}(\mu_j) = \mu_j, j \in J\}$ with $\mu \leq \bigvee_{j \in J} \mu_j$, there exists a finite subset J_0 of J such that $\mu \leq \bigvee_{j \in J_0} \mathrm{cl}_R^{\lambda}(\mu_j)$ (resp. $\mu \,\overline{\wedge}\, (\bigvee_{j \in J_0} \mathrm{cl}_{\Phi}^{\lambda}(\mu_j)) \in \ell$).
(3) μ is said to be fuzzy nearly approximation compact (resp. fuzzy nearly ideal approximation compact) if for any family $\{\mu_j \in I^X : \ \mathrm{int}_R^{\lambda}(\mu_j) = \mu_j, j \in J\}$ with $\mu \leq \bigvee_{j \in J} \mu_j$, there exists a finite subset J_0 of J such that $\mu \leq \bigvee_{j \in J_0} \mathrm{int}_R^{\lambda}(\mathrm{cl}_R^{\lambda}(\mu_j))$ (resp. $\mu \,\overline{\wedge}\, (\bigvee_{j \in J_0} \mathrm{int}_R^{\lambda}(\mathrm{cl}_{\Phi}^{\lambda}(\mu_j))) \in \ell$).

The fuzzy approximation space (X, R) (resp. The fuzzy ideal approximation space (X, R, ℓ)) will be called fuzzy approximation compact, fuzzy almost approximation compact, fuzzy nearly approximation compact (resp. fuzzy ideal approximation compact, fuzzy almost ideal approximation compact, fuzzy nearly ideal approximation compact) if we replaced μ with $\overline{1}$.

It is clear that:

$$\begin{tikzcd} \mathit{compact} \arrow[r] \arrow[d] & \mathit{almost\ compact} \arrow[r] \arrow[d] & \mathit{nearly\ compact} \arrow[d] \\ \mathit{ideal\ compact} \arrow[r] & \mathit{almost\ ideal\ compact} \arrow[r] & \mathit{nearly\ ideal\ compact} \end{tikzcd}$$

If $\ell = \{\overline{0}\}$, then
fuzzy approximation compact (fuzzy almost approximation compact, fuzzy nearly approximation compact) and fuzzy ideal approximation compact (fuzzy almost ideal approximation compact, fuzzy nearly ideal approximation compact) respectively, are equivalent.

Here is an example for both of Definition 5.1 and Definition 5.2.

Example 5.1. Let R be a fuzzy relation on a set $X = \{a, b, c, d\}$ defined as follows.

R	a	b	c	d
a	1	1	0	0
b	1	1	0	0
c	0	0	1	0.3
d	0	0	0.3	1

(1) Assume that $\lambda = \{1, 1, 0.6, 0.6\}$. Then, $\lambda_R = \{1, 1, 0.6, 0.6\} = \lambda$ and $(\lambda_R)^c = \{0, 0, 0.4, 0.4\} = \lambda^c$.
For any $\nu_j \in I^X$ with $\{1, 1, 0, 0\} \leq \nu_j \leq \{1, 1, 0.6, 0.6\}$, we get that
$\mathrm{int}^{\lambda}_R(\nu_j) = (\nu_j)_R \wedge \lambda_R = \nu_j \wedge \lambda = \nu_j$. That is, for any $\mu \in I^X$ with $\mathrm{int}^{\lambda}_R(\mu) = \mu$, we can choose a family of these fuzzy sets ν_j such that $\mathrm{int}^{\lambda}_R(\nu_j) = \nu_j$ and $\mathrm{cl}^{\lambda}_R(\nu_j) \leq \mu$ whenever we choose only the fuzzy sets w_j so that $\{1, 1, 0.4, 0.4\} \leq w_j \leq \{1, 1, 0.6, 0.6\}$, which satisfy that $\mathrm{cl}^{\lambda}_R(w_j) = w_j$. Thus, we get for all $\mu \in I^X$ with $\{1, 1, 0.4, 0.4\} \leq \mu \leq \{1, 1, 0.6, 0.6\}$, a family of fuzzy sets w_j so that $\{1, 1, 0.4, 0.4\} \leq w_j \leq \{1, 1, 0.6, 0.6\}$ for which

$$\mu = \bigvee_{j \in J} \{w_j \ : \ \mathrm{int}^{\lambda}_R(w_j) = w_j, \ \mathrm{cl}^{\lambda}_R(w_j) \leq \mu\}.$$

Note that: the condition for μ is satisfied only if μ is a special fuzzy set but not for all $\mu \in I^X$. For example, $\mu = \{1, 1, 0.2, 0.2\}$ with $\mathrm{int}^{\lambda}_R(\mu) = \mu$, there is no $\{\nu_j\}_{j \in J}$ satisfying the condition for μ, where $\{1, 1, 0.4, 0.4\} \leq \mathrm{cl}^{\lambda}_R(\nu_j) \nleq \mu$ for all ν_j with $\mathrm{int}^{\lambda}_R(\nu_j) = \nu_j$.

Hence, (X, R) is not fuzzy regular approximation space. Similarly, we can show that the fuzzy ideal approximation space (X, R, ℓ) is not fuzzy ideal regular space whenever $\ell = \{\overline{0}\}$.

(2) Assuming $\lambda = \{1, 0, 0, 0\}$ and the same fuzzy relation R on X. Then, $\lambda_R = \overline{0}$ and $(\lambda_R)^c = \overline{1}$. For any family of $\nu_j \in I^X$, we get that $\mathrm{int}^{\lambda}_R(\nu_j) = \overline{0}$, and thus for any fuzzy set $\mu \in I^X$, we get that $\mu \in I^X$ is satisfying directly the definition

$$\mu = \bigvee_{j \in J} \{\nu_j \ : \ \mathrm{int}^{\lambda}_R(\nu_j) = \nu_j, \ \mathrm{cl}^{\lambda}_R(\nu_j) \leq \mu\}.$$

That is, (X, R) is a fuzzy regular approximation space associated with this $\lambda \in I^X$. Similarly, we can show that the fuzzy ideal approximation space (X, R, ℓ) is a fuzzy ideal regular space whenever $\ell = \{\overline{0}\}$.

(3) Moreover, associated with $\lambda = \{1, 1, 0.6, 0.6\}$ and the same fuzzy relation R on X, we can prove that $\eta = \{1, 1, 0.5, 0.5\}$ is a fuzzy approximation compact set, where $\eta \leq \bigvee_{j \in J} \nu_j$ and η itself is one of these fuzzy sets ν_j. That is, for every fuzzy cover from these ν_j, $j \in J$ of η, there is a finite subcover η itself as a cover of η. In addition, η is fuzzy ideal approximation compact set if we restricted the fuzzy ideal ℓ on X to be only $\{\overline{0}\}$.
The other two cases of compactness are easily shown by choosing the fuzzy cover as the same family of sets w_j with $\{1, 1, 0.4, 0.4\} \leq w_j \leq \{1, 1, 0.6, 0.6\}$.

Theorem 5.1. *Let (X, R, ℓ) be fuzzy almost ideal approximation compact and fuzzy ideal regular. Then, X is a fuzzy ideal approximation compact space.*

Proof. Assume a family $\{\mu_j \in I^X : \mathrm{int}_R^\lambda(\mu_j) = \mu_j,\ j \in J\}$ with $\overline{1} = \bigvee_{j\in J} \mu_j$.

By fuzzy ideal regularity of X, then for each $\mathrm{int}_R^\lambda(\mu_j) = \mu_j$, we have

$$\mu_j = \bigvee_{j_k \in J_K} \{\mu_{j_k} :\ \mathrm{int}_R^\lambda(\mu_{j_k}),\ \mathrm{cl}_\Phi^\lambda(\mu_{j_k}) \leq \mu_j\}.$$

Hence, $\overline{1} = \bigvee_{j\in J}(\bigvee_{j_k\in J_K} \mu_{j_k})$. Since X is fuzzy almost ideal approximation compact, then there exists a finite index subset $J_0 \times J_K$ of $J \times J$ such that

$$\overline{1} \,\overline{\wedge}\, (\bigvee_{j\in J_0}(\bigvee_{j_k\in J_K} \mathrm{cl}_\Phi^\lambda(\mu_{j_k}))) \in \ell.$$

Since for each $j \in J_0$, we have $\bigvee_{j_k\in J_K} \mathrm{cl}_\Phi^\lambda(\mu_{j_k}) \leq \mu_j$, then we get that

$$\overline{1} \,\overline{\wedge}\, (\bigvee_{j\in J_0}(\bigvee_{j_k\in J_K} \mathrm{cl}_\Phi^\lambda(\mu_{j_k}))) \geq \overline{1} \,\overline{\wedge}\, (\bigvee_{j\in J_0} \mu_j).$$

Therefore, $\overline{1} \,\overline{\wedge}\, (\bigvee_{j\in J_0} \mu_j) \in \ell$, and thus (X, R, ℓ) is fuzzy ideal approximation compact. □

Theorem 5.2. *Let (X, R, ℓ) be fuzzy nearly ideal approximation compact and fuzzy ideal regular. Then, X is a fuzzy nearly ideal approximation compact.*

Proof. Similar to the proof of Theorem 5.1. □

Theorem 5.3. *Let $f : (X, R, \ell_1) \to (Y, R^*, \ell_2)$ be injective fuzzy approximation continuous mapping between two fuzzy ideal approximation spaces associated with $\lambda \in I^X$, $\mu \in I^Y$ respectively and $\nu \in \ell_1 \Longrightarrow f(\nu) \in \ell_2\ \forall \nu \in I^X$, and $\eta \in I^X$ is a fuzzy ideal approximation compact set. Then, $f(\eta)$ is fuzzy ideal approximation compact as well.*

Proof. Let $\{\xi_j \in I^Y : \mathrm{int}_{R^*}^\mu(\xi_j) = \xi_j, j \in J\}$ be a family with $f(\eta) \leq \bigvee_{j\in J} \xi_j$.

By fuzzy approximation continuity of f, $\mathrm{int}_R^\lambda(f^{-1}(\xi_j)) = f^{-1}(\xi_j)$ and $\eta \leq \bigvee_{j\in J} f^{-1}(\xi_j)$.

By fuzzy ideal approximation compactness of η, there exists a finite subset J_0 of J such that $\eta \,\overline{\wedge}\, (\bigvee_{j\in J_0}(f^{-1}(\xi_j))) \in \ell_1$.

Since $\nu \in \ell_1 \Longrightarrow f(\nu) \in \ell_2\ \forall \nu \in I^X$, then $f(\eta \,\overline{\wedge}\, (\bigvee_{j\in J_0}(f^{-1}(\xi_j)))) \in \ell_2$. From f is injective, then $f(\eta \,\overline{\wedge}\, (\bigvee_{j\in J_0}(f^{-1}(\xi_j)))) = f(\eta) \,\overline{\wedge}\, (\bigvee_{j\in J_0}(\xi_j))$. Thus,

$$f(\eta) \,\overline{\wedge}\, (\bigvee_{j\in J_0}(\xi_j)) \in \ell_2.$$

Hence, $f(\eta)$ is fuzzy ideal approximation compact. □

6. Conclusion

Let X be a non empty set and $\mathcal{G} \subseteq I^X$. Then, $\mathcal{G}$ is a fuzzy grill on X ([4]) iff $\ell(\mathcal{G}) = \{\mu \in I^X : \mu \notin \mathcal{G}\}$ is a fuzzy ideal on X, and conversely; let X be a non empty set and $\ell \subseteq I^X$. Then, ℓ is a fuzzy ideal on X iff $\mathcal{G}(\ell) = \{\mu^c \in I^X : \mu \in \ell\}$ is a fuzzy grill on X. If we defined the fuzzy approximation separation axioms or the fuzzy approximation connectedness, or the fuzzy approximation compactness using the notion of fuzzy grill, it will be the same definitions and results as given using the notion of fuzzy ideal from the correspondence between fuzzy ideals and fuzzy grills.

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